GREEN'S THEOREM FOR GENERALIZED FRACTIONAL DERIVATIVES

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Abstract. We study three types of generalized partial fractional operators. An extension of Green's theorem, by considering partial fractional derivatives with more general kernels, is proved. New results are obtained, even in the particular case when the generalized operators are reduced to the standard partial fractional derivatives and fractional integrals in the sense of Riemann–Liouville or Caputo.

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1. Introduction

In 1828, the English mathematician George Green (1793-1841), who up to his forties was working as a baker and a miller, published an essay where he introduced a formula connecting the line integral around a simple closed curve with a double integral. Within years, this result turned out to be useful in many fields of mathematics, physics and engineering [4, 6, 15, 17]. Generalizations of Green's theorem have chosen different directions, and are known as the Kelvin–Stokes and the Gauss–Ostrogradsky theorems.

In this paper, in contrast with previous works, we want to state a Green's theorem for generalized partial fractional derivatives. Notions of generalized fractional derivatives were introduced in [1, 8], and then developed in [11, 12]. A fractional version of the Green theorem has been already showed for Riemann–Liouville integrals and Caputo derivatives [18], and for fractional operators in the sense of Jumarie [3]. However, generalized fractional operators have never been considered. Our result may be useful in the theory of fractional calculus (see, e.g., [7, 9, 14, 16]), in particular

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for the two-dimensional fractional calculus of variations, where the derivation of Euler–Lagrange equations uses, as a key step in the proof, Green's theorem [3, 5, 10, 13].

The paper is organized as follows. In Section 2 a common review of ordinary and partial generalized fractional calculus is given. Our results are then formulated and proved in Section 3: we show the two-dimensional integration by parts formula for generalized Riemann–Liouville partial fractional integrals (Theorem 3.1) and Green's theorem for generalized partial fractional derivatives (Theorem 3.2).

2. Basic Notions

In this section we give definitions of generalized ordinary and partial fractional operators. By the choice of a certain kernel, these operators can be reduced to the standard fractional integrals and derivatives. For more on the subject, we refer the reader to [1, 2, 8, 11, 12].

2.1. **Generalized fractional operators.**

Definition 2.1 (Generalized fractional integral). The operator K_P^α is given by

$$\left(K_P^\alpha f\right)(t) := p\int_a^t k_\alpha(t,\tau)f(\tau)d\tau + q\int_t^b k_\alpha(\tau,t)f(\tau)d\tau,$$

where $P=\langle a,t,b,p,q\rangle$ is the *parameter set* (p-set for brevity), $t\in[a,b]$, p,q are real numbers, and $k_\alpha(t,\tau)$ is a kernel which may depend on α. The operator K_P^α is referred as the *operator* K (K-op for simplicity) of order α and p-set P.

Theorem 2.1 (Theorem 2.3 of [11]). *Let k_α be a difference kernel, i.e., $k_\alpha(t,\tau)=k_\alpha(t-\tau)$ and $k_\alpha\in L_1([a,b])$. Then, $K_P^\alpha : L_1([a,b])\to L_1([a,b])$ is well defined, bounded and linear operator.*

The K-op reduces to the classical left or right Riemann–Liouville fractional integral (see, e.g., [7, 14]) for a suitably chosen kernel $k_\alpha(t,\tau)$ and p-set P. Indeed, let $k_\alpha(t-\tau)=\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$. If $P=\langle a,t,b,1,0\rangle$, then

$$\left(K_P^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau =: \left({}_aI_t^\alpha f\right)(t)$$

is the left Riemann–Liouville fractional integral of order α; if $P = \langle a, t, b, 0, 1\rangle$, then

$$\left(K_P^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} f(\tau) d\tau =: \left({}_t I_b^\alpha f\right)(t)$$

is the right Riemann–Liouville fractional integral of order α.

Definition 2.2 (Generalized Riemann–Liouville derivative). Let P be a given parameter set. The operator A_P^α, $0 < \alpha < 1$, is defined for functions f such that $K_P^{1-\alpha} f \in AC\left([a,b]\right)$ by $A_P^\alpha := \frac{d}{dt} \circ K_P^{1-\alpha}$, where D denotes the standard derivative. We refer to A_P^α as *operator* A (A-op) of order α and p-set P.

Definition 2.3 (Generalized Caputo derivative). Let P be a given parameter set. The operator B_P^α, $\alpha \in (0,1)$, is defined for functions f such that $f \in AC\left([a,b]\right)$ by $B_P^\alpha := K_P^{1-\alpha} \circ \frac{d}{dt}$ and is referred as the *operator* B (B-op) of order α and p-set P.

Let $k_{1-\alpha}(t-\tau) = \frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha}$, $\alpha \in (0,1)$. If $P = \langle a, t, b, 1, 0\rangle$, then

$$\left(A_P^\alpha f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau =: \left({}_a D_t^\alpha f\right)(t)$$

is the standard left Riemann–Liouville fractional derivative of order α while

$$\left(B_P^\alpha f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau =: \left({}_a^C D_t^\alpha f\right)(t)$$

is the standard left Caputo fractional derivative of order α; if $P = \langle a, t, b, 0, 1\rangle$, then

$$-\left(A_P^\alpha f\right)(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} f(\tau) d\tau =: \left({}_t D_b^\alpha f\right)(t)$$

is the standard right Riemann–Liouville fractional derivative of order α while

$$-\left(B_P^\alpha f\right)(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau - t)^{-\alpha} f'(\tau) d\tau =: \left({}_t^C D_b^\alpha f\right)(t)$$

is the standard right Caputo fractional derivative of order α.

2.2. **Generalized partial fractional operators.** Let α be a real number from the interval $(0,1)$, $\Delta_n = [a_1,b_1]\times\cdots\times[a_n,b_n]$, $n \in \mathbb{N}$, be a subset of $\mathbb{R}^n$, $\mathbf{t} = (t_1,\ldots,t_n)$ be a point in Δ_n and $\mathbf{p} = (p_1,\ldots,p_n)$, $\mathbf{q} = (q_1,\ldots,q_n) \in \mathbb{R}^n$. Generalized partial fractional integrals and derivatives are a natural generalization of the corresponding one-dimensional generalized fractional integrals and derivatives.

Definition 2.4 (Generalized partial fractional integral). Let function $f = f(t_1,\ldots,t_n)$ be continuous on the set Δ_n. The generalized partial Riemann–Liouville fractional integral of order α with respect to the ith variable t_i is given by

$$\left(K^{\alpha}_{P_{t_i}} f\right)(\mathbf{t}) := p_i \int\limits_{a_i}^{t_i} k_\alpha(t_i,\tau) f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau + q_i \int\limits_{t_i}^{b_i} k_\alpha(\tau,t_i) f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau,$$

where $P_{t_i} = \langle a_i, t_i, b_i, p_i, q_i\rangle$. We refer to $K^{\alpha}_{P_{t_i}}$ as the *partial operator K* (partial K-op) of order α and p-set P_{t_i}.

Definition 2.5 (Generalized partial Riemann–Liouville derivatives). Let $P_{t_i} = \langle a_i, t_i, b_i, p_i, q_i\rangle$ and $K^{1-\alpha}_{P_{t_i}} f \in C^1(\Delta_n)$. The generalized partial Riemann–Liouville fractional derivative of order α with respect to the ith variable t_i is given by

$$\begin{aligned}(A^{\alpha}_{P_{t_i}} f)(\mathbf{t}) &:= \left(\frac{\partial}{\partial t_i} \circ K^{1-\alpha}_{P_{t_i}} f\right)(\mathbf{t}) \\ &= \frac{\partial}{\partial t_i}\left(p_i \int\limits_{a_i}^{t_i} k_{1-\alpha}(t_i,\tau) f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \right. \\ &\qquad \left. + q_i \int\limits_{t_i}^{b_i} k_{1-\alpha}(\tau,t_i) f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau\right).\end{aligned}$$

The operator $A^{\alpha}_{P_{t_i}}$ is referred as the *partial operator A* (partial A-op) of order α and p-set P_{t_i}.

Definition 2.6 (Generalized partial Caputo derivative). Let $P_{t_i} = \langle a_i, t_i, b_i, p_i, q_i\rangle$ and $f \in C^1(\Delta_n)$. The generalized partial Caputo fractional

derivative of order α with respect to the ith variable t_i is given by

$$\begin{aligned}\left(B^{\alpha}_{P_{t_i}} f\right)(\mathbf{t}) &:= \left(K^{1-\alpha}_{P_{t_i}} \circ \frac{\partial}{\partial t_i} f\right)(\mathbf{t}) \\ &= p_i \int\limits_{a_i}^{t_i} k_{1-\alpha}(t_i,\tau)\frac{\partial}{\partial \tau} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \\ &\qquad + q_i \int\limits_{t_i}^{b_i} k_{1-\alpha}(\tau,t_i)\frac{\partial}{\partial \tau} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau\end{aligned}$$

and is referred as the *partial operator* B (partial B-op) of order α and p-set P_{t_i}.

Similarly as in the one-dimensional case [1, 11, 12], the generalized partial operators K, A and B here introduced give the standard partial fractional integrals and derivatives for particular kernels and p-sets. The left- and right-sided Riemann–Liouville partial fractional integrals with respect to the ith variable t_i are obtained by choosing the kernel

$$k_\alpha(t_i,\tau) = \frac{1}{\Gamma(\alpha)}(t_i-\tau)^{\alpha-1}$$

and p-sets $L_{t_i} = \langle a_i, t_i, b_i, 1, 0\rangle$ and $R_{t_i} = \langle a_i, t_i, b_i, 0, 1\rangle$, respectively:

$$\begin{aligned}\left({}_{a_i}I^{\alpha}_{t_i} f\right)(\mathbf{t}) &= \left(K^{\alpha}_{L_{t_i}} f\right)(\mathbf{t}) \\ &= \frac{1}{\Gamma(\alpha)} \int\limits_{a_i}^{t_i} (t_i-\tau)^{\alpha-1} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau,\end{aligned}$$

$$\begin{aligned}\left({}_{t_i}I^{\alpha}_{b_i} f\right)(\mathbf{t}) &= \left(K^{\alpha}_{R_{t_i}} f\right)(\mathbf{t}) \\ &= \frac{1}{\Gamma(\alpha)} \int\limits_{t_i}^{b_i} (\tau-t_i)^{\alpha-1} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau.\end{aligned}$$

The standard left- and right-sided partial Riemann–Liouville and Caputo fractional derivatives with respect to the ith variable t_i are obtained with the choice of kernel $k_{1-\alpha}(t_i,\tau) = \frac{1}{\Gamma(1-\alpha)}(t_i-\tau)^{-\alpha}$: if $P_{t_i} = \langle a_i, t_i, b_i, 1, 0\rangle$,

then

$$\begin{aligned}\left({}_{a_i}D^{\alpha}_{t_i}f\right)(\mathbf{t}) &= \left(A^{\alpha}_{P_{t_i}}f\right)(\mathbf{t})\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t_i}\int\limits_{a_i}^{t_i}(t_i-\tau)^{-\alpha}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau\end{aligned}$$

and

$$\begin{aligned}\left({}^{C}_{a_i}D^{\alpha}_{t_i}f\right)(\mathbf{t}) &= \left(B^{\alpha}_{P_{t_i}}f\right)(\mathbf{t})\\ &= \frac{1}{\Gamma(1-\alpha)}\int\limits_{a_i}^{t_i}(t_i-\tau)^{-\alpha}\frac{\partial}{\partial\tau}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau;\end{aligned}$$

if $P_{t_i}=\langle a_i,t_i,b_i,0,1\rangle$, then

$$\begin{aligned}\left({}_{t_i}D^{\alpha}_{b_i}f\right)(\mathbf{t}) &= -\left(A^{\alpha}_{P_{t_i}}f\right)(\mathbf{t})\\ &= -\frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t_i}\int\limits_{t_i}^{b_i}(\tau-t_i)^{-\alpha}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau\end{aligned}$$

and

$$\begin{aligned}\left({}^{C}_{t_i}D^{\alpha}_{b_i}f\right)(\mathbf{t}) &= -\left(B^{\alpha}_{P_{t_i}}f\right)(\mathbf{t})\\ &= -\frac{1}{\Gamma(1-\alpha)}\int\limits_{t_i}^{b_i}(\tau-t_i)^{-\alpha}\frac{\partial}{\partial\tau}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau.\end{aligned}$$

Remark 2.1. In Definitions 2.4, 2.5 and 2.6, all the variables, except t_i, are kept fixed. That choice of fixed values determines a function $f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}:[a_i,b_i]\to\mathbb{R}$ of one variable t_i:

$$f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)=f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n).$$

By Definitions 2.1, 2.2, 2.3 and 2.4, 2.5, 2.6, we have

$$\begin{aligned}\left(K^{\alpha}_{P_{t_i}}f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}\right)(t_i) &= \left(K^{\alpha}_{P_{t_i}}f\right)(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),\\ \left(A^{\alpha}_{P_{t_i}}f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}\right)(t_i) &= \left(A^{\alpha}_{P_{t_i}}f\right)(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),\\ \left(B^{\alpha}_{P_{t_i}}f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}\right)(t_i) &= \left(B^{\alpha}_{P_{t_i}}f\right)(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n).\end{aligned}$$

Therefore, as in the classical integer order case, computation of partial generalized fractional operators is reduced to the computation of one-variable generalized fractional operators.

3. Green's Theorem for Generalized Fractional Derivatives

Definition 3.1 (Dual p-set). Let $P_{t_i} = \langle a_i, t_i, b_i, p_i, q_i \rangle$, $i \in \mathbb{N}$. We denote by $P_{t_i}^*$ the p-set $P_{t_i}^* = \langle a_i, t_i, b_i, q_i, p_i \rangle$ and call it the dual of P_{t_i}.

Theorem 3.1 (Generalized 2D Integration by Parts). *Let $\alpha \in (0,1)$, $P_{t_i} = \langle a_i, t_i, b_i, p_i, q_i \rangle$ be a parameter set, and k_α be a difference kernel, i.e., $k_\alpha(t_i, \tau) = k_\alpha(t_i - \tau)$ such that $k_\alpha \in L_1([0, b_i - a_i])$, $i = 1, 2$. If $f, g, \eta_1, \eta_2 \in C(\Delta_2)$, then the generalized partial fractional integrals satisfy the following identity:*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} \left[g(\boldsymbol{t}) \left(K_{P_{t_1}}^{\alpha} \eta_1 \right)(\boldsymbol{t}) + f(\boldsymbol{t}) \left(K_{P_{t_2}}^{\alpha} \eta_2 \right)(\boldsymbol{t}) \right] dt_2 dt_1$$

$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2} \eta_1(\boldsymbol{t}) \left[\left(K_{P_{t_1}^*}^{\alpha} g \right)(\boldsymbol{t}) \right] + \eta_2(\boldsymbol{t}) \left[\left(K_{P_{t_2}^*}^{\alpha} f \right)(\boldsymbol{t}) \right] dt_2 dt_1,$$

where $P_{t_i}^*$ is the dual of P_{t_i}, $i = 1, 2$.

Proof. Define

$$F_1(\mathbf{t}, \tau) := \begin{cases} |p_1 k_\alpha(t_1 - \tau)| \cdot |g(\mathbf{t})| \cdot |\eta_1(\tau, t_2)| & \text{if } \tau \le t_1 \\ |q_1 k_\alpha(\tau - t_1)| \cdot |g(\mathbf{t})| \cdot |\eta_1(\tau, t_2)| & \text{if } \tau > t_1 \end{cases}$$

for all $(\mathbf{t}, \tau) \in [a_1, b_1] \times [a_2, b_2] \times [a_1, b_1]$ and

$$F_2(\mathbf{t}, \tau) := \begin{cases} |p_2 k_\alpha(t_2 - \tau)| \cdot |f(\mathbf{t})| \cdot |\eta_2(t_1, \tau)| & \text{if } \tau \le t_2 \\ |q_2 k_\alpha(\tau - t_2)| \cdot |f(\mathbf{t})| \cdot |\eta_2(t_1, \tau)| & \text{if } \tau > t_2 \end{cases}$$

for all $(\mathbf{t}, \tau) \in [a_1, b_1] \times [a_2, b_2] \times [a_2, b_2]$. Since f, g and η_i, $i = 1, 2$, are continuous functions on Δ_2, they are bounded on Δ_2. Hence, there exist real numbers $C_1, C_2, C_3, C_4 > 0$ such that

$$|f(\mathbf{t})| \le C_1, \quad |g(\mathbf{t})| \le C_2, \quad |\eta_1(\mathbf{t})| \le C_3, \quad |\eta_2(\mathbf{t})| \le C_4,$$

for all $\mathbf{t} \in \Delta_2$. Therefore,

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_1}^{b_1} F_1(\mathbf{t},\tau)dt_1dt_2d\tau + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_2}^{b_2} F_2(\mathbf{t},\tau)dt_2d\tau dt_1 \\
&\quad= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{\tau}^{b_1}|p_1k_\alpha(t_1-\tau)|\cdot|g(\mathbf{t})|\cdot|\eta_1(\tau,t_2)|\,dt_1\right.\right. \\
&\qquad\left.\left.+\int_{a_1}^{\tau}|q_1k_\alpha(\tau-t_1)|\cdot|g(\mathbf{t})|\cdot|\eta_1(\tau,t_2)|\,dt_1\right)dt_2\right)d\tau \\
&\qquad+\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{\tau}^{b_2}|p_2k_\alpha(t_2-\tau)|\cdot|f(\mathbf{t})|\cdot|\eta_2(t_1,\tau)|\,dt_2\right.\right. \\
&\qquad\left.\left.+\int_{a_2}^{\tau}|q_2k_\alpha(\tau-t_2)|\cdot|f(\mathbf{t})|\cdot|\eta_2(t_1,\tau)|\,dt_2\right)d\tau\right)dt_1 \\
&\quad\le C_2C_3\left[\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{\tau}^{b_1}|p_1k_\alpha(t_1-\tau)|\,dt_1\right.\right.\right. \\
&\qquad\left.\left.\left.+\int_{a_1}^{\tau}|q_1k_\alpha(\tau-t_1)|\,dt_1\right)dt_2\right)d\tau\right] \\
&\qquad+C_1C_4\left[\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{\tau}^{b_2}|p_2k_\alpha(t_2-\tau)|\,dt_2\right.\right.\right. \\
&\qquad\left.\left.\left.+\int_{\tau}^{b_2}|q_2k_\alpha(\tau-t_2)|\,dt_2\right)d\tau\right)dt_1\right] \\
&\quad\le C_2C_3\left[\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{0}^{b_1-a_1}|p_1k_\alpha(u_1)|\,du_1\right.\right.\right. \\
&\qquad\left.\left.\left.+\int_{0}^{b_1-a_1}|q_1k_\alpha(u_1)|\,du_1\right)dt_2\right)d\tau\right] \\
&\qquad+C_1C_4\left[\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{0}^{b_2-a_2}|p_2k_\alpha(u_2)|\,du_2\right.\right.\right. \\
&\qquad\left.\left.\left.+\int_{0}^{b_2-a_2}|q_2k_\alpha(u_2)|\,du_2\right)d\tau\right)dt_1\right] \\
&\quad= C_2C_3\left(|p_1|+|q_1|\right)\|k_\alpha\|(b_2-a_2)(b_1-a_1) \\
&\qquad+C_1C_4\left(|p_2|+|q_2|\right)\|k_\alpha\|(b_2-a_2)(b_1-a_1) \\
&\quad<\infty.
\end{aligned}$$

Hence, we can use Fubini's theorem to change the order of integration in the iterated integrals:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\left(K^{\alpha}_{P_{t_1}}\eta_1\right)(\mathbf{t})+f(\mathbf{t})\left(K^{\alpha}_{P_{t_2}}\eta_2\right)(\mathbf{t})\right]dt_2dt_1$$

$$
\begin{aligned}
&= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\left(p_1\int_{a_1}^{t_1} k_\alpha(t_1-\tau)\eta_1(\tau,t_2)d\tau\right.\right.\\
&\quad \left. + q_1\int_{t_1}^{b_1} k_\alpha(\tau-t_1)\eta_1(\tau,t_2)d\tau\right)\\
&\quad + f(\mathbf{t})\left(p_2\int_{a_2}^{t_2} k_\alpha(t_2-\tau)\eta_2(t_1,\tau)d\tau\right.\\
&\quad \left.\left. + q_2\int_{t_2}^{b_2} k_\alpha(\tau-t_2)\eta_2(t_1,\tau)d\tau\right)\right] dt_2dt_1\\
&= \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1}\eta_1(\tau,t_2)\left(p_1\int_{\tau}^{b_1} k_\alpha(t_1-\tau)g(\mathbf{t})dt_1\right.\right.\\
&\quad \left.\left. + q_1\int_{a_1}^{\tau} k_\alpha(\tau-t_1)g(\mathbf{t})dt_1\right)d\tau\right)dt_2\\
&\quad + \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\eta_2(t_1,\tau)\left(p_2\int_{\tau}^{b_2} k_\alpha(t_2-\tau)f(\mathbf{t})dt_2\right.\right.\\
&\quad \left.\left. + q_2\int_{a_2}^{\tau} k_\alpha(\tau-t_2)f(\mathbf{t})dt_2\right)d\tau\right)dt_1\\
&= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta_1(\tau,t_2)\left(K^{\alpha}_{P_{t_1^*}}g\right)(\tau,t_2)dt_2d\tau\\
&\quad + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta_2(t_1,\tau)\left(K^{\alpha}_{P_{t_2^*}}f\right)(t_1,\tau)d\tau dt_1.
\end{aligned}
$$

□

We are now in conditions to state and prove the main result of the paper: the Green theorem for generalized fractional derivatives.

THEOREM 3.2 (Generalized Green's Theorem). *Let $0<\alpha<1$ and $f,g,\eta\in C^1(\Delta_2)$. Let k_α be a difference kernel, i.e., $k_\alpha(t_i,\tau)=k_\alpha(t_i-\tau)$ such that $k_\alpha\in L_1([0,b_i-a_i])$, $i=1,2$, and $K^{1-\alpha}_{P^*_{t_1}}g, K^{1-\alpha}_{P^*_{t_2}}f\in C^1(\Delta_2)$. Then, the following formula holds:*

$$
\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\boldsymbol{t})\left(B^{\alpha}_{P_{t_1}}\eta\right)(\boldsymbol{t})+f(\boldsymbol{t})\left(B^{\alpha}_{P_{t_2}}\eta\right)(\boldsymbol{t})\right]dt_2dt_1\\
&\qquad = -\int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\boldsymbol{t})\left[\left(A^{\alpha}_{P^*_{t_1}}g\right)(\boldsymbol{t})+\left(A^{\alpha}_{P^*_{t_2}}f\right)(\boldsymbol{t})\right]dt_2dt_1\\
&\qquad\qquad + \oint_{\partial\Delta_2}\eta(\boldsymbol{t})\left[\left(K^{1-\alpha}_{P^*_{t_1}}g\right)(\boldsymbol{t})dt_2-\left(K^{1-\alpha}_{P^*_{t_2}}f\right)(\boldsymbol{t})dt_1\right].
\end{aligned}
$$

P r o o f. By the definition of generalized partial Caputo fractional derivative, Theorem 3.1, and the standard Green's theorem, one has

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\left(B^{\alpha}_{P_{t_1}}\eta\right)(\mathbf{t})+f(\mathbf{t})\left(B^{\alpha}_{P_{t_2}}\eta\right)(\mathbf{t})\right]dt_2dt_1\\
&\quad=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\left(K^{1-\alpha}_{P_{t_1}}\frac{\partial}{\partial t_1}\eta\right)(\mathbf{t})+f(\mathbf{t})\left(K^{1-\alpha}_{P_{t_2}}\frac{\partial}{\partial t_2}\eta\right)(\mathbf{t})\right]dt_2dt_1\\
&\quad=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[\frac{\partial}{\partial t_1}\eta(\mathbf{t})\left(K^{1-\alpha}_{P^*_{t_1}}g\right)(\mathbf{t})+\frac{\partial}{\partial t_2}\eta(\mathbf{t})\left(K^{1-\alpha}_{P^*_{t_2}}f\right)(\mathbf{t})\right]dt_2dt_1\\
&\quad=-\int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left[\frac{\partial}{\partial t_1}\left(K^{1-\alpha}_{P_{t^*_1}}g\right)(\mathbf{t})+\frac{\partial}{\partial t_2}\left(K^{1-\alpha}_{P_{t^*_2}}f\right)(\mathbf{t})\right]dt_2dt_1\\
&\qquad+\oint_{\partial\Delta_2}\eta(\mathbf{t})\left[\left(K^{1-\alpha}_{P^*_{t_1}}g\right)(\mathbf{t})dt_2-\left(K^{1-\alpha}_{P^*_{t_2}}f\right)(\mathbf{t})dt_1\right].
\end{aligned}$$

□

COROLLARY 3.1. *Let $0<\alpha<1$ and $f,g,\eta\in C^1(\Delta_2)$. If $\left({}_{t_1}I^{1-\alpha}_{b_1}g\right)(\boldsymbol{t})$ and $\left({}_{t_2}I^{1-\alpha}_{b_2}f\right)(\boldsymbol{t})$ are continuously differentiable on the rectangle Δ_2, then*

$$\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\left({}^{C}_{a_1}D^{\alpha}_{t_1}\eta\right)(\mathbf{t})+f(\mathbf{t})\left({}^{C}_{a_2}D^{\alpha}_{t_2}\eta\right)(\mathbf{t})\right]dt_2dt_1\\
&\qquad=\int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left[\left({}_{t_1}D^{\alpha}_{b_1}g\right)(\mathbf{t})+\left({}_{t_2}D^{\alpha}_{b_2}f\right)(\mathbf{t})\right]dt_2dt_1\\
&\qquad\qquad+\oint_{\partial\Delta_2}\eta(\mathbf{t})\left[\left({}_{t_1}I^{1-\alpha}_{b_1}g\right)(\mathbf{t})dt_2-\left({}_{t_2}I^{1-\alpha}_{b_2}f\right)(\mathbf{t})dt_1\right].
\end{aligned}$$

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